Ricardo, New York Math Circle, NY; Titu Zvonaru, Comănesti, Romania (jointly with) Neculai Stanciu, "George Emil Palade School," Buzău, Romania, and the proposer.

- **5318:** *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

Prove that $(1+x)^x \le 1+x^2$ for $0 \le x \le 1$.

Solution 1 by Arkady Alt, San Jose, California, USA.

The inequality $(1+x)^x \le 1+x^2, 0 \le x \le 1$ immediately follows from the Bernoulli inequality:

$$(1+t)^\alpha \ge 1+\alpha t, t > -1, a \ge 1. \qquad \mathbf{(1)}$$

Indeed, for $0 < x \le 1 \iff \dfrac{1}{x} \ge 1,$ and by **(1)** we have

$$\left(1+x^2\right)^{\frac{1}{x}} \ge 1+x^2 \cdot \frac{1}{x} = 1+x \iff 1+x^2 \ge (1+x)^x.$$

For $x=0$ the original inequality is obvious.

Another way to prove the inequality $(1+x)^x \le 1+x^2$ is based on using the Weighted AM-GM Inequality: $u^p v^q \le pu+qv$ where $u,v,p,q \ge 0$ and $p+q=1$.

Indeed, for $u=1+x, v=1,\ p=x,\ q=1-x$ we have

$$(1+x)^x \cdot 1^{1-x} \le (1+x)\,x + 1\cdot(1-x) \iff (1+x)^x \le 1+x^2.$$

$(1+x)^a \le 1+ax,$ where $x > -1$ and $0 \le a \le 1.$ **(1)**

Applying inequality **(1)** to $a=x$ we obtain $(1+x)^x \le 1+x^2$.

Solution 2 by Albert Stadler, Herrliberg, Switzerland

We have equalityfor $x=0$ and $x=1$.

We assume that $0 < x < 1$. We expand $(1+x)^x$ into a binomial series and get

$$\begin{aligned}(1+x)^x &= \sum_{n=0}^{\infty}\binom{x}{n}x^n = 1+x^2+\sum_{j=1}^{\infty}\left(\frac{x(x-1)\cdots(x-2j+1)}{(2j)!}x^{2j}+\frac{x(x-1)\cdots(x-2j)}{(2j+1)!}x^{2j+1}\right)\\ &= 1+x^2-\sum_{j=1}^{\infty}\underbrace{\frac{x(1-x)\cdots(2j-1-x)}{(2j)!}x^{2j}}_{>0}\underbrace{\left(1-\frac{2j-x}{2j+1}x\right)}_{>0} < 1+x^2.\end{aligned}$$

Solution 3 by Michael Brozinsky, Central Islip, NY

Consider $g(u)=(1+u)^{\dfrac{1}{u}}$ on $(0,1)$. It is decreasing since

$$g'(u) = (1+u)^{\dfrac{1}{u}} \cdot \frac{\left(\dfrac{u}{1+u}-\ln(1+u)\right)}{u^2}$$